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Department of Mathematics

MATH 2055 Tutorial 4 (Oct 7)

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1. Write down the negations of the following statements.

   (a) $\forall \epsilon > 0$, $\exists N$ such that $\forall n > N$, $|x_n - x| < \epsilon$

   Solution: $\exists \epsilon > 0$, such that $\forall N_0$, $\exists n > N_0$, $|x_n - x| > \epsilon$

   (b) $\exists N$, such that $\forall n > N$, $\forall \epsilon > 0$, $|x_n - x| < \epsilon$

   Solution: $\forall N_0$, $\exists n > N_0$, $\exists \epsilon > 0$, $|x_n - x| > \epsilon$

   (c) $\forall M$, $\exists N$, such that $\forall n > N$, $|x_n - x| > M$

   Solution: $\exists M$, $\forall N_0$, $\exists n > N_0$, $|x_n - x| < M$

2. For a pair of positive numbers a and b, define sequences a_n and b_n respectively as

$$a_1 = a, \qquad b_1 = b$$
$$a_{n+1} = \frac{a_n + b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n}$$

   Prove that $a_n \geq a_{n+1} \geq b_{n+1} \geq b_n$ for $n \geq 2$, and they have the same limit.

   Solution:
   $\forall \alpha, \beta \in \mathbb{R}^+$,

   $(\sqrt{\alpha} + \sqrt{\beta})^2 \geq 0 \Longrightarrow (\alpha + \beta)/2 \geq \sqrt{\alpha\beta}$

   $\therefore \forall n \geq 2$, $a_n \geq b_n$

$$b_{n+1} - b_n = \sqrt{a_n b_n} - b_n = \frac{b_n(a_n - b_n)}{\sqrt{a_n b_n} + b_n} \geq 0$$
$$a_n - a_{n+1} = a_n - \frac{a_n + b_n}{2} = \frac{a_n - b_n}{2} \geq 0$$

$\therefore$ $\{a_n\}_{n\geq 2}$ is decreasing and bounded below by b_2 and $\{b_n\}_{n\geq 2}$ is increasing and bounded above by a_2

$\therefore$ $\{a_n\}$ and $\{b_n\}$ are convergent

$\because$ $a_{n+1} = \frac{a_n+b_n}{2}$,

$$\implies \lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n}{2}$$

$$\implies \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

3. Let $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots \frac{1}{n+n}$, $\forall n \in \mathbb{N}$ show that (x_n) is convergent.

Solution:
$\forall n \geq 1$,

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{(n+1)+(n)} + \frac{1}{(n+1)+(n+1)} - \frac{1}{n+1} \\ &= \frac{1}{2(2n+1)(n+1)} > 0 \end{aligned}$$

$\therefore$ $\{x_n\}$ is increasing

$$\begin{aligned} x_n &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \\ &\leq \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1} \\ &= \frac{n}{n+1} < 1 \end{aligned}$$

$\therefore$ $\{x_n\}$ is bounded above

$\Longrightarrow \{x_n\}$ is convergent

Remarks: $x_1 = 1/2 \Longrightarrow \lim_{n\to\infty} x_n \geq 1/2$
but $\lim_{n\to\infty} \frac{1}{n+i} = 0$ for all natural number i
It means that we cannot calculate the limit terms by terms

4. Let (b_n) be a bounded sequence of non-negative numbers and r be any number such that $0 \leq r < 1$

   Define $s_n = b_1 r + b_2 r^2 + \cdots + b_n r^n$

   prove that s_n is convergent.

   Solution:
   $\forall n \geq 1$,

   $s_{n+1} - s_n = b_{n+1} r^{n+1} \geq 0$

   $\Longrightarrow \{s_n\}$ is increasing

   $\because \{b_n\}$ is bounded

   $\exists M$ such that $\forall n$, $b_n \leq M$

$$\begin{aligned} s_n &\leq Mr + Mr^2 + \cdots + Mr^n \\ &= Mr\left(\frac{1-r^n}{1-r}\right) \\ &< \frac{Mr}{1-r} \end{aligned}$$

   $\therefore \{s_n\}$ is bounded above

   $\therefore \{s_n\}$ converges

5. Given that (x_n) is increasing sequence , $x_{n+1} - x_n < 1$, and $x_1 \geq 0$

If $x_n({x_n}^2 - (2n + \frac{n+1}{n})x_n + 2(n+1)) > 0$

Prove that (x_n) is convergent.

Solutions:
We only need to prove boundedness

$0 \leq x_n({x_n}^2 - (2n + \frac{n+1}{n}x_n + 2(n+1)) = x_n(x_n - \frac{n+1}{n})(x_n - 2n)$

$\Longrightarrow 0 < x_n < \frac{n+1}{n}$ or $x_n > 2n$

idea: the "speed" of increasing of 2n larger than that of x_n

If $\exists n_0$ such that $x_{n_0} > 2n_0$

let m be the smallest integer bigger than $x_{n_0} - 2n_0$, then $2n_0 > x_{n_0} - m$

as $x_{n+1} - x_n < 1$, we have $x_{n_0+m} < x_{n_0} + m$

$2(n_0 + m) > x_{n_0} - m + 2m = x_{n_0} + m > x_{n_0+m}$

$\Longrightarrow 0 < x_{n_0+m} < \dfrac{(n_0+m)+1}{n_0+m}$

but $x_{n_0+m} > x_n > 2n_0 > \dfrac{(n_0+m)+1}{n_0+m}$ which lead to contradiction

$\therefore \forall n$, $0 < x_n < \frac{n+1}{n} < 2$

$\therefore \{x_n\}$ converges